

# Phenomenology of magnetically active superconductors

by

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**Summary:** The thermodynamical phenomenological theory of magnetically active superconducting materials and magnetic-superconducting heterostructures is presented. The materials may exhibit arbitrarily strong anisotropy, parametric or structural. Exact anisotropic similarity transformations are found for both the continuum and layered hybrid systems.

**Keywords:** Ginzburg-Landau theory, variational principles, superconductors, magnetic order, anisotropy, heterostructures, hybrid systems

## 1 Introduction

A formulation of thermodynamical theory of magnetically active, anisotropic materials admitting coexistence of the superconductivity and magnetic order has been proposed in [1]. The theory is based on the Ginzburg-Landau approach extended to multi-component order parameters and the states of thermodynamic quasi-equilibrium far from the superconducting phase transition. The field equations are derived under assumption of the  $U(1)$  gauge invariance. The nonlinearity with respect to the order parameter, magnetic field, and other physical quantities is assumed reasonably general, without any particular restrictions of its functional form.

In the present paper we give an improved version of that theory complemented with the exact anisotropic similarity transformations for layered hybrid systems [23].

## 2 Background

A material medium is considered magnetically active if its magnetic properties differ (substantially) from those of the physical vacuum. Although the superconductivity itself is a magnetic phenomenon, the superconducting materials need not be magnetically active. This circumstance justifies disregarding the magnetic activity of a superconducting material in heuristic or didactic considerations, but not in a more complete theory. Due to orbital polarizability and/or spin degrees of freedom, the effects of magnetic activity of superconducting materials are practically always present, and in many instances significant. In particular

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it concerns the heterostructures which involve interactions between superconducting and ferromagnetic components [6, 18]. One may also notice that, on the background of some controversial views [4, 8] concerning coexistence of superconductivity and ferromagnetism, materials which combine these phenomena were found experimentally [3, 11, 14, 15]. Moreover, it seems that, in accordance with earlier theoretical predictions, the superconductivity of such materials is magnetically mediated [12].

In order to describe the superconducting state of condensed matter, the GINZBURG-LANDAU theory [9] introduces a phenomenological quantity, called the order parameter, represented by a complex scalar field  $\psi(x)$  with properties quite analogous to the microscopic quantum-mechanical single-particle wave function. Contrary, however, to the microscopic wave function, the GL order parameter describes the macroscopic condensate of the supercurrent carriers and enters as an independent variable into the thermodynamical free energy functional. This approach turned out to be extremely successful.

In the original GL paper the functional was given a simplest form suitable for description of the states of a superconducting material near the phase transition. In this form, the theory was confirmed afterwards by GORKOV [10] through rederivation from the microscopic theory. Its range of applicability turned out, however, severely restricted by the condition of proximity to the transition point.

To cope with broader ranges of real situations, the GL theory has been extended in various directions, including multicomponent order parameters, more elaborate forms of the free energy functional, time-dependent phenomena, and heterostructures. Any such modification, as long as it respects the idea of a (possibly generalized) superconducting order parameter as a thermodynamical variable, is referred to as belonging to the framework of *extended* Ginzburg-Landau phenomenology [21, 22, 23].

The theory under consideration admits arbitrary (in a reasonable sense) functional forms of the GL thermodynamical potential. We intend to include in this way also the thermodynamical states which are far from the superconducting phase transition. We admit multiple order parameters, anisotropy, and material inhomogeneities, but, for simplicity, we retain the assumptions of stationarity, thermodynamical quasi-equilibrium, and charge neutrality.

The assumption of charge neutrality refers to the resultant electric charge density of all charged components, such as the superconducting carriers (Cooper pairs), normal electrons, and the ionic substrate. It restricts our considerations to the case of bulk superconductivity in good (quasi-metallic) superconductors. To include "bad" superconductors and interface effects one would need a further extension of the model, which is beyond the scope of the present paper and will be given elsewhere.

We shall focus our attention on superconducting materials which can be (a) strongly anisotropic, (b) inhomogeneous at the macro- and micro-scale, and (c) magnetically active. Below we shall comment briefly on the above marked specific assumptions.

(a) The idea of isotropic superconductor, due to its simplicity and heuristic power, attracts much attention as a tool in theoretical investigations. It suffices here to note examples of most outstanding and successful classic theories of superconductivity such as Ginzburg-Landau, Bardeen-Cooper-Schrieffer, and Abrikosov's ones, each of them based on an appropriate isotropic model. On the other hand, the real superconducting materials exhibit, as a rule, some anisotropy of physical properties. In particular, the anisotropy of high  $T_c$  cuprate oxides is rather high [5, 7]: e.g. the ratio of penetration depths  $\lambda_c/\lambda_{ab}$  for  $\text{YBa}_2\text{Cu}_3\text{O}_{7-y}$

and  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+y}$  lies in the range  $5\div 7$  and  $50\div 200$ , respectively; it corresponds to estimated values of the ratio of flux line tensions  $U_c/U_{ab}$  ranging from 20 to 40000.

(b) To describe heterogeneous superconducting materials or devices, the theory must admit material inhomogeneities of various – macroscopic, microscopic, and intermediate – scales of length. The inhomogeneities can be smooth or abrupt, technological or natural. Many high  $T_c$  superconducting materials exhibit a distinct layered structure at the atomistic level. The spacings between atomic layers, such as Cu-O planes in cuprate-oxide superconductors, can be relatively large as compared with the in-plane distances what, besides the structural anisotropy, can result in a natural micro-scale inhomogeneity of those materials in the out-of-plane direction. The LAWRENCE-DONIACH model [13] and its generalization [22, 23] can be considered an extreme case of such a natural micro-inhomogeneity. On the other hand, due to the recent progress in nanotechnology it is possible to fabricate diverse nano-scale technological heterostructures.

(c) Apart from magnetic activity originating from physical factors, we found it also expedient to introduce its equivalent at the level of theoretical modelling. We refer here to the anisotropic similarity transformations - a useful tool in the study of anisotropic superconductors, discussed in Section 8.

### 3 Notation

Whenever appropriate, we use implicit tensor notation with suppressed indices; the indices are exposed only when otherwise the expressions would be equivocal or illegible. For instance, a point in the physical space (as well as the corresponding material point when the distinction makes no difference) is denoted simply by  $x$ , which is to be developed to  $(x^1, x^2, x^3)$  in typical 3D situation.

Basically, the standard notation from recent literature on phenomenology of superconductivity [20] is employed throughout the paper. The effective mass  $m^*$  of the supercurrent carriers is tensorial; in the case of transversally isotropic superconducting materials (e.g. layered oxides, in good approximation) the relevant mass tensor components are  $m_{ab}$  (in-plane) and  $m_c$  (out-of-plane).

The complex conjugate of the order parameter  $\psi$  is denoted by  $\bar{\psi}$ . The symbols  $\partial$  and  $\nabla$  are employed in a somewhat non-standard way. The partial derivative operator is denoted by  $\partial$ , while the symbol  $\nabla$  is reserved for the electromagnetic  $U(1)$ -covariant derivative. Both operators  $\partial$  and  $\nabla$  act from the left. We found it expedient to introduce also the conjugate operators  $\bar{\partial}$  and  $\bar{\nabla}$  which act from the right.

Due to a bug in the routine translating from TeX to PDF, in the final printed form of the reference [1] a repeated typographic error occurred. The error consisted in the fusion of two consecutive short overbars into a longer one. It resulted in erroneous replacement of the expressions  $\bar{\psi} \bar{\partial}$  with  $\overline{\psi \partial}$  and  $\bar{\psi} \bar{\nabla}$  with  $\overline{\psi \nabla}$ . The present text contains terms counteracting those faults.

## 4 The reference model

To fix the point of departure for construction of extended models we start from the reference model which, excepting the details of notation, almost exactly coincides with the original GL model [9]. It is also convenient to develop the notation adapted in this paper.

1. The superconducting state of the superconductor is described with the aid of the order parameter represented by a single-component complex scalar function  $\psi = \psi(x)$ .

The gauge transformation for this parameter is defined as

$$\begin{cases} A \rightarrow A + \partial\Lambda, \\ \psi \rightarrow e^{\frac{ie^*}{\hbar c}\Lambda}\psi, \\ \bar{\psi} \rightarrow \bar{\psi}e^{-\frac{ie^*}{\hbar c}\Lambda}, \end{cases} \quad (1)$$

where  $e^*$  denotes the electric charge of the supercurrent carriers. The corresponding covariant derivatives  $\nabla$  and  $\bar{\nabla}$  are given by

$$\nabla = \partial - \frac{ie^*}{\hbar c}A, \quad (2)$$

and

$$\bar{\nabla} = \bar{\partial} + \frac{ie^*}{\hbar c}A. \quad (3)$$

2. The free energy of the superconductor is given by the functional

$$F[T; \psi, A] = \int d^3x F_S \quad (4)$$

with an appropriate free energy density  $F_S$  of the superconducting state.

3. The free energy density  $F_S$  equals

$$F_S = F_N + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{\hbar^2}{2m^*} \bar{\psi} \bar{\nabla} \nabla \psi, \quad (5)$$

where  $F_N$  refers to the normal state and  $m^*$  represents the effective mass of the supercurrent carriers. The coefficients  $\alpha$  and  $\beta$  are given as

$$\alpha = a(T - T_c), \quad a = \text{const}, \quad \beta = \text{const}. \quad (6)$$

The normal state free energy density  $F_N$  does not depend on the order parameter.

The expression (5) ensures the already required invariance with respect to the gauge transformations (1).

## 5 Local first order functional

The reference model sketched in the previous section relies upon many intentional simplifications, including isotropy, homogeneity, and proximity to the N-S transition point. Some of these drawbacks are direct consequences of the particular form of the free energy density  $F_S$  and can be eliminated simply by admitting, in place of (5), more general functionals defined on the same space of the field variables. In this way one can successfully take into account

many realistic features of superconductors. As examples one can mention such properties as high anisotropy or more subtle thermodynamical behaviour of high  $T_c$  superconductors [16, 5, 20].

The range of applicability of such models is, however, severely restricted by the singlet nature of the order parameter. From the microscopic point of view a singlet order parameter is justified, first of all, for superconductors with dominating isotropic  $s$ -pairing. Also other types of pairing can be treated in this way, provided that a single pairing mode, corresponding to a one-dimensional representation of the internal symmetry group, is strongly dominating.

Many superconducting materials require, however, multiple order parameters. One can mention here the heavy fermion unconventional superconductors [17] which exhibit anisotropic spin triplet  $p$ -pairing, possibly mixed with spin singlet  $s$ -pairing. Another typical example is delivered by the coexistence of spin singlet  $s$ - and  $d$ -pairings in orthorombic high  $T_c$  materials [19].

In the present section we shall briefly discuss an extension of the GL phenomenology which avoids the above mentioned drawbacks.

1. First of all we modify the assumption stated in item 1 of Section 1. In place of the singlet order parameter, represented in the reference model by a complex-valued scalar field, a more general multi-component order parameter is introduced. For brevity, this order parameter will also be denoted by the single symbol  $\psi = \psi(x)$ . In geometrical interpretation the admissible values of  $\psi$  will be considered as organized into a differential manifold. Whenever a particular component will be referred to, an indexed symbol like  $\psi^A = \psi^A(x)$  will be used and interpreted as an appropriate – real or complex – coordinate in the corresponding local map. The index  $A$  will take integer values from 1 to  $N$ , with arbitrary fixed  $N$ . The real dimension of the order parameter manifold may, due to the presence of complex-valued coordinates, differ from  $N$ .

2. From the physical point of view the multiplet order parameter represents generally a mixture of charged fields. In contrast to the reference model, such a mixture can not be characterized by a single scalar charge. To characterize the coupling of the multiplet order parameter  $\psi$  with the electromagnetic field we must specify an adequate gauge transformation. To that end, in place of the scalar charge  $e^*$  we introduce a real symmetric matrix

$$e^* = (e^*_{AB}) \tag{7}$$

such that the gauge transformation is defined by the corresponding matrix reinterpretation of the old formula (1). In this way also the gauge covariant derivatives (2) and (3) retain their graphic form with reinterpreted meaning.

In general, the charge matrix  $e^*$  need not be diagonal. However, being real and symmetric, it can be diagonalized by an appropriate orthogonal transformation of the order parameters  $\psi^A$ . The eigenvalue 0, if present, corresponds to electrically neutral subspace of order parameters which carries no supercurrent; in spite of this property such order parameters can seriously affect the thermodynamical properties of a superconducting system. The remaining eigenvalues of  $e^*$  determine the spectrum of charges of supercurrent carriers.

In most cases the supercurrent carriers are experimentally identified as electrically equivalent to pairs of electrons or holes. Then the spectrum of the matrix  $e^*$  simplifies to at most 3 eigenvalues:  $\pm 2e$  and 0.

3. We accept the assumption formulated in item 2 of Section 4, so that the free energy functional retains the local form (4). We, however, modify item 3.: instead of the particular form (5) we assume

$$F_S = F[\psi, A] = F(T, x, \psi, \bar{\psi}, \partial\psi, \bar{\psi}\bar{\partial}, A, \partial A). \quad (8)$$

The explicit dependence on  $x$  is intended to reflect natural or artificial inhomogeneities of the materials, such as inhomogeneous doping, structure defects, etc., including possible heterostructural properties of the system. Implicit dependence on some external control parameters (such as mechanical deformation) is admitted; the implicit parameters can also depend on the position  $x$ .

Apart from reasonable mathematical properties and the gauge invariance, no particular restrictions will, in general, be imposed upon the function (8). The symbol  $e^*$  stands here for the matrix (7).

4. The requirement of gauge invariance results in the following relations

$$\frac{\partial F}{\partial\psi}e^*\psi - \bar{\psi}e^*\frac{\partial F}{\partial\bar{\psi}} + \frac{\partial F}{\partial(\partial\psi)}e^*\partial\psi - \bar{\psi}\bar{\partial}e^*\frac{\partial F}{\partial(\bar{\psi}\bar{\partial})} = 0, \quad (9)$$

$$\frac{\partial F}{\partial A} + \frac{i}{\hbar c}\left(\frac{\partial F}{\partial(\partial\psi)}e^*\psi - \bar{\psi}e^*\frac{\partial F}{\partial(\bar{\psi}\bar{\partial})}\right) = 0, \quad (10)$$

and

$$\frac{\partial F}{\partial(\partial_j A_k)} + \frac{\partial F}{\partial(\partial_k A_j)} = 0 \quad (11)$$

to be satisfied identically by the function (8).

The conditions (10) and (11) can be integrated in a straightforward way. As a result, one obtains the functional dependence

$$F_S = F[\psi, A] = F(T, x, \psi, \bar{\psi}, \nabla\psi, \bar{\psi}\bar{\nabla}, B) \quad (12)$$

with the vector potential  $A$  entering indirectly through the gauge covariant derivatives  $\nabla, \bar{\nabla}$  and the (gauge invariant) magnetic induction field  $B = \text{curl } A$ . The gauge invariance conditions for the function (12) reduce to

$$\frac{\partial F}{\partial\psi}e^*\psi - \bar{\psi}e^*\frac{\partial F}{\partial\bar{\psi}} + \frac{\partial F}{\partial(\nabla\psi)}e^*\nabla\psi - \bar{\psi}\bar{\nabla}e^*\frac{\partial F}{\partial(\bar{\psi}\bar{\nabla})} = 0. \quad (13)$$

5. Taking into account the functional dependence (12), the functional derivatives of  $F_S$  with respect to the field variations  $\delta\psi$ ,  $\delta\bar{\psi}$ , and  $\delta A$  can be represented in the form

$$\frac{\delta F}{\delta\psi} = \frac{\partial F}{\partial\psi} - \nabla\frac{\partial F}{\partial(\nabla\psi)}, \quad (14)$$

$$\frac{\delta F}{\delta\bar{\psi}} = \frac{\partial F}{\partial\bar{\psi}} - \frac{\partial F}{\partial(\bar{\psi}\bar{\nabla})}\bar{\nabla}, \quad (15)$$

and

$$\frac{\delta F}{\delta A} = \frac{i}{\hbar c}\left(\bar{\psi}e^*\frac{\partial F}{\partial(\bar{\psi}\bar{\nabla})} - \frac{\partial F}{\partial(\nabla\psi)}e^*\psi\right) + \text{curl } \frac{\partial F}{\partial B}. \quad (16)$$

6. In consequence, one obtains the generalized Ginzburg-Landau equation

$$\frac{\partial F}{\partial \bar{\psi}} - \frac{\partial F}{\partial(\bar{\psi} \bar{\nabla})} \bar{\nabla} = 0, \quad (17)$$

and the magnetic field equation

$$\text{curl } H = \frac{4\pi}{c} j \quad (18)$$

with the intensity of magnetic field

$$H = 4\pi \frac{\partial F}{\partial B} \quad (19)$$

and the supercurrent density

$$j = -\frac{i}{\hbar} (\bar{\psi} e^* \frac{\partial F}{\partial(\bar{\psi} \bar{\nabla})} - \frac{\partial F}{\partial(\nabla \psi)} e^* \psi). \quad (20)$$

7. Taken literally, the above equations are derived for sufficiently smooth fields. As a rule, such degree of differentiability can not be guaranteed at abrupt interfaces, in particular at the heterostructural ones. However, the integral form of those equations is valid under much weaker differentiability conditions, sufficient for the derivation of boundary conditions. The standard procedure applied to the equation (14) leads to the boundary conditions, which in index notation take the form of continuity equation across the interface

$$\frac{\partial F^I}{\partial(\bar{\psi}^A \bar{\nabla}_k)} n^k = \frac{\partial F^{II}}{\partial(\bar{\psi}^A \bar{\nabla}_k)} n^k \quad (21)$$

where  $n^k$  represents the normal vector of the surface demarcating the regions I and II. In consequence of this equation the normal component of the supercurrent (20) is continuous.

## 6 Layered hybrid model

The free energy functional for higher-grade hybrid model [23] has the following form

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_S + \int g(\mathbf{B}) d^3x. \quad (22)$$

The last term represents the contribution of magnetic field to the free energy. In the standard particular case of magnetically inactive material the function  $g(\mathbf{B})$  reduces to  $\mathbf{B}^2/8\pi$ . The term  $\mathcal{F}_0$  describes the normal state, while  $\mathcal{F}_S$  the superconducting one. The superconducting term is composed of two parts:

$$\mathcal{F}_S = \mathcal{F}_P + \mathcal{F}_J, \quad (23)$$

where the part

$$\mathcal{F}_P = s \sum_n \int dxdy F_n \quad (24)$$

describes the contribution of atomic planes, while  $\mathcal{F}_J$  corresponds to interplanar Josephson's bonds.  $F_n$  has the Ginzburg-Landau form (5), with 2D operator  $\nabla$  and the coefficients dependent, in general, on the plane  $n$ .

The form of the functional  $\mathcal{F}_P$  already ensures its invariance with respect to the gauge transformation (1), understood this time in the 2D sense.

The Josephson term in the free energy functional for the hybrid model of grade K has the form

$$\mathcal{F}_J = s \sum_n \sum_q \int dx dy \varepsilon_{qn}. \quad (25)$$

The quantity  $\varepsilon_{qn}$  denotes the energy of J-link between  $n$ -th and  $(n+q)$ -th planes and is given by

$$\varepsilon_{qn} = \frac{1}{2} \{ \zeta_q (|\psi_n|^2 + |\psi_{n+q}|^2) - \gamma_q (\bar{\psi}_n \psi_{n+q} e^{-ip_{qn}} + c.c.) \}, \quad (26)$$

with the coupling parameters  $\zeta_q$  and  $\gamma_q$  vanishing for  $q > K$ . The exponent  $p_{qn}$  is defined by the formula

$$p_{qn} = \frac{e^*}{\hbar c} \int_{ns}^{(n+q)s} A_z dz. \quad (27)$$

## 7 Structured order parameters, nonlocal and higher order models

In the extension of GL phenomenology discussed in Section 5 we restricted our attention to superconductors which can be described by local first order functionals with multiplet order parameters. Such a restriction greatly simplifies the theoretical considerations. On the other hand, it eliminates some physical phenomena which are known to be present in superconductors. It concerns, in particular, the nonlocal effects and nonparabolic dispersion curves. Such phenomena can, however, be taken into account by admitting yet more general – nonlocal and/or higher order – free energy functionals.

The microscopic theories of superconducting materials of complex atomic and electronic structure indicate various possible mechanisms of superconductivity [2]. To cope with the diversity of those mechanisms, the phenomenological description can be further generalized by introducing the structured order parameter. Instead of the customary order parameter represented by a (possibly multicomponent) function  $\psi(x)$ , we introduce a generalized one, represented by a function of  $\psi(x, \xi)$ , where the new variable  $\xi$  runs over a certain internal space  $\Sigma$ . The space  $\Sigma$  serves as a mathematical stage on which the internal structure of superconducting carriers (such as Cooper pairs) appears.

The space  $\Sigma$  depends on the material under consideration and, parametrically, on the temperature, doping, and other control parameters such as external mechanical deformation. In the case of uniform material samples the space  $\Sigma$  is the same for all material points; in the case of heterostructures and inhomogeneous material samples it can become  $x$ -dependent.

The concept of structured order parameter offers tighter bound between the phenomenology and the microscopic theories. Apart from more accurate quantitative description it allows also to study possible topological transitions related to changes in the deep topological structure of the space  $\Sigma$ . The detailed discussion of those effects is, however, beyond the scope of the present paper.



## 8 Anisotropic similarity and scaling

Assume now that the order parameter space is endowed with a linear structure and consider a linear transformation  $x \rightarrow x', \psi \rightarrow \psi', A \rightarrow A'$  of the following form

$$\begin{cases} x' = Qx, \\ \psi' = R\psi, \\ A' = \tilde{Q}A, \end{cases} \quad (28)$$

where  $Q$ ,  $R$ , and  $\tilde{Q}$  are linear matrices; the matrices are real except  $R$  which can be, totally or partially, complex. The transformation rule

$$\partial' = \partial Q^{-1} \quad (29)$$

is an immediate consequence of (28).

1. The transformation (28) in its general form violates the  $U(1)$  gauge structure. If we request that, under (28), the depart gauge covariant derivative be transformed into the target gauge covariant derivative,  $\nabla\psi \rightarrow \nabla'\psi'$ , the transformation (28) must be adequately restricted. It follows that the relation  $\tilde{Q} = Q^{-1}$  must hold. Such a transformation will be called a (gauge covariant) anisotropic similarity. The resulting transformation rules for the gauge covariant derivatives of  $\psi(x)$  and  $\bar{\psi}(x)$  take the form

$$\nabla'\psi' = RQ^{-1T}\nabla\psi, \quad \bar{\psi}'\nabla' = \bar{\psi}\bar{\nabla}Q^{-1}R^+, \quad (30)$$

where the superscripts  $T$  and  $+$  stand for the matrix transposition and Hermitian conjugation, respectively.

2. Consider two material systems, say the dashed and the undashed ones, described by the thermodynamical functionals (4) with the densities  $F'(\cdot)$  and  $F(\cdot)$ , respectively. We shall say that those systems are anisotropically similar if

$$F'(T, x, \psi, \bar{\psi}, \partial\psi, \bar{\psi}\bar{\partial}, A, \partial A) = F(T, x', \psi', \bar{\psi}', \partial'\psi', \bar{\psi}'\bar{\partial}', A', \partial' A'). \quad (31)$$

Due to the gauge covariance this is equivalent to

$$F'(T, x, \psi, \bar{\psi}, \nabla\psi, \bar{\psi}\bar{\nabla}, B) = F(T, x', \psi', \bar{\psi}', \nabla'\psi', \bar{\psi}'\bar{\nabla}', B'). \quad (32)$$

3. To justify this conclusion let us examine the behaviour of the field equations (17 – 20) under the transformations (28). Taking into account the transformation rules (28 - 30) we obtain from eqn. (32)

$$\frac{\partial F'}{\partial \psi'} = R^{-1+} \frac{\partial F}{\partial \bar{\psi}}, \quad (33)$$

$$\frac{\partial F'}{\partial(\bar{\psi}'\bar{\nabla}')} = R^{-1+} Q \frac{\partial F}{\partial(\bar{\psi}\bar{\nabla})}. \quad (34)$$

In consequence, from the expression (20) one obtains

$$j' = Qj. \quad (35)$$

Further,

$$B' = (\det Q)^{-1} Q B \quad (36)$$

and

$$H' = (\det Q) Q^{-1} H. \quad (37)$$

The relations (33 – 37) guarantee the covariance of the field equations (17) and (18) with respect to the anisotropic similarity transformations.

In the layered hybrid model a straightforward calculation shows that the energy of the Josephson's coupling (26) is invariant with respect to transformation

$$\begin{cases} \psi'_n = (-1)^n \psi_n, \\ \mathbf{A}' = \mathbf{A}, \\ \gamma'_q = (-1)^q \gamma_q, \end{cases} \quad (38)$$

with all the remaining quantities kept fixed.

This transformation switches between systems with all the constants  $\gamma_q$  transformed according to the parity of  $q$ : the constants remain identical for even  $q$ 's and change sign for odd  $q$ 's; in particular  $\gamma'_1 = -\gamma_1$  and  $\gamma'_2 = \gamma_2$ . At the same time the order parameter  $\psi_n$  is transformed by the sign-alternating factor; in particular the states of uniform order parameter are transformed into alternating ones, and *vice versa*.

In consequence the energy of the Josephson's coupling is invariant with respect to transformation (38). This invariance extends to the Josephson part of the total free energy functional (25). Two hybrid systems related by the transformation (38) have identical ground state energies, excitation energy spectra, and all the thermodynamical properties, including  $T_c$ . The shapes of the corresponding ground states are, however, substantially different.

The usefulness of the above transformations stems from the fact that they allow to obtain a solution valid for one system from a solution valid for another – anisotropically similar – system. Analogous transformations, known as anisotropic scaling, have been defined in reference [5]. However, although very useful as a tool for getting approximate estimations, they do not imply exact invariance. In opposition to that, our anisotropic similarity transformations are exact. In consequence, transformed exact solutions for the depart system are always guaranteed to be also exact solutions for the target system.

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